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Solutions of Bianchi-VI_0 space-times with perfect fluid

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Abstract : Spatially homogeneous Bianchi VI_0 space-time with perfect fluid distribution is considered and it is shown that the field equations are solvable for any arbitrary cosmic scale function. A one-parameter family of solutions for a particular form of a cosmic scale function is presented which would essentially represent empty universe for large time.

Keywords : Perfect fluid, Bianchi VI_0

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The spatially homogeneous Bianchi spaces I–IX are useful tools for constructing cosmological models to describe the behaviour of the universe at early stages of its evolution. Bianchi-VI_0 spaces are of particular interest since they are sufficiently complex, while at the same time, they are a simple generalization of Bianchi I spaces. Since long a great deal of theoretical work has been done to build Bianchi VI_0 cosmological models by solving Einstein's field equations associated with different matter distributions. Ellis and MacCallum [1] obtained solutions of Einstein's field equations in the case of a stiff-fluid. Collins [2] and Ruban [3] presented some exact solutions of this type for perfect fluid distribution satisfying the specific equation of state. Dunn and Tupper [4] investigated a class of Bianchi type VI_0 perfect fluid cosmological models associated with electromagnetic fields. Lorentz [5] generalized the dust model given by Ellis and MacCallum [1]. Roy and Singh [6] derived some exact solutions of Einstein-Maxwell equations representing a free gravitational field of the magnetic type with perfect fluid and incident magnetic field. Ribeiro and Sanyal [7] studied spatially homogeneous Bianchi-VI_0 models containing a viscous fluid in the presence of an axial magnetic field. In the above work, the perfect fluid considered necessarily obeys specific equation of state. Following Hajj-Bautras [8,9] Shri Ram [10] presented an algorithm for generating a new exact perfect fluid solutions at Einstein's field-equations for spatially homogeneous cosmological models of Bianchi type VI_0 without choosing any equation of state.

Recently, Mazumdar [11] has shown that the field equations for LRS Bianchi-I space-time filled with a perfect fluid are solvable for any arbitrary cosmic scale function. In this paper, we present a procedure to obtain new perfect fluid solutions of Bianchi type VI_0. For a particular form of a cosmic function a one-parameter family of solutions is presented. Solutions can be added to rare perfect fluid solutions of type VI_0 not satisfying the equation of state. The physical properties of solutions are also discussed.

The metric for the Bianchi type VI_0 class of models is taken to be of the form [12]

$$ds^2 = dt^2 - A^2dx^2 - B^2e^{2mx}dy^2 - C^2e^{-2mx}dz^2 \tag{1}$$

where A, B, C are functions of cosmic time 't' and 'm' is a constant parameter. The Einstein's field equations are [13]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -T_{\mu\nu} \tag{2}$$

Here $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ the Ricci tensor, R the scalar curvature.

For a perfect fluid distribution, the energy-momentum tensor $T_{\mu\nu}$ is of the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu} \tag{3}$$

where u^μ is 4-velocity vector, p the pressure and ρ the mass-energy density.

In comoving coordinates, the field equations to be considered are

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{m^2}{A^2} = -p \tag{4}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{m^2}{A^2} = -p \tag{5}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{m^2}{A^2} = -p \tag{6}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{m^2}{A^2} = \rho \tag{7}$$

$$\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) = 0 \tag{8}$$

A dot denotes differentiation with respect to 't'. Equation (8) on integration, yields

$$B = KC \tag{9}$$

where K is a constant, without loss of any generality we can take $K = 1$. Eliminating p from (4) and (5) we get

$$\left(\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A}\right) + \frac{\dot{B}}{B}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) + \frac{2m^2}{A^2} = 0 \tag{10}$$

Making the scale transformation

$$dt = AB^2 d\tau \tag{11}$$

the equation (10) becomes

$$\left(\frac{B''}{B} - \frac{B'^2}{B^2}\right) - \left(\frac{A''}{A} - \frac{A'^2}{A^2}\right) + 2m^2 B^4 = 0 \tag{12}$$

A dash denotes differentiation with respect to τ. The first integral of (12) is

$$\frac{A'}{A} - \frac{B'}{B} = 2m^2\left(\int B^4 d\tau + C_1\right) \tag{13}$$

C_1 being integration constant. If we define the function $F(\tau)$ by

$$F(\tau) = 2m^2\left(\int B^4 d\tau + C_1\right) \tag{14}$$

Then equation (13), on integration, yields

$$A = C_2 B \exp\int F(\tau) d\tau, \tag{15}$$

C_2 being another integration constant.

Particular exact solutions to the Einstein's equations (4)–(7) can be found using the formula (15) if B is given as an explicit functions of τ, A can be computed from (15) The energy density and pressure are determined by (4), (7) and (11) Note that in this case the integration of Einstein's equations reduces to the integration of (15). A number of solutions to (15) could be generated by assuming different functional forms to B. Here we obtain a physically realistic solution by assuming

$$B = \tau^n, \tag{16}$$

where 'n' is an arbitrary parameter. Inserting (16) into (15) we obtain

$$A = C_2 \tau^n \exp\left[\frac{2m^2 \tau^{4n+2}}{(4n+1)(4n+2)} + C_1 \tau\right]. \tag{17}$$

The metric of the solutions can be written in the form

$$ds^2 = \tau^{2n} e^{-\left[\frac{-m}{(4n+1)(4n+2)} \tau^{4n+2} + C_1 \tau\right]}\left(+ \tau^{4n} d\tau^2 - dx^2\right.$$
$$\left. - \tau^{2n}\left(e^{-2mx} dy^2 + e^{2mx} d\tau^2\right)\right). \tag{18}$$

For the metric (18), pressure and energy density are obtained from (6) and (7) as

$$p = \frac{1}{\tau^{6n}\left[C_2 \exp\left\{2m^2\left(\frac{\tau^{4n+2}}{(4n+1)(4n+2)} + C_1\tau\right)\right\}\right]^2}$$

$$\left[4nm^2\left\{\frac{\tau^{4n}}{(4n+1)} + \frac{C_1}{\tau}\right\} + \frac{3n^2}{\tau^2} + \frac{2n}{\tau} - m^2\tau^{4n}\right] \tag{19}$$

$$\rho = \frac{1}{\tau^{6n}\left[C_2 \exp\left\{2m^2\left(\frac{\tau^{4n+2}}{(4n+1)(4n+2)} + C_1\tau\right)\right\}\right]^2}$$

$$\left[\left(\frac{4nm^2}{4n+1} - m^2\right)\tau^{4n} + \frac{3n^2}{\tau^2} + \frac{4nm^2C_1}{\tau}\right] \tag{20}$$

for the physical reality of the solution we take $n \geq 0$

We discuss the properties of the shear tensor It has been pointed out by Collins and Wainwright [14] that the shear tensor $\sigma_{\mu\nu}$ plays an important role in general relativistic cosmological and stellar models. The shear tensor arises in the decomposition of four -velocity vector of the fluid, i.e

$$u_{\mu;\nu} = -u'_\mu u_\nu + w_{\mu\nu} + \sigma_{\mu\nu} + \theta h_{\mu\nu}/3$$

$$u'_\mu = u_{\mu;\nu}u^\nu, \qquad u'_\mu u^\mu = 0$$

$$w_{\mu\nu} = u_{[\mu;\nu]} + u'_{[\mu}u_{\nu]}, \qquad w_{\mu\nu}u^\nu = 0$$

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \qquad h_{\mu\nu}u^\nu = 0$$

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} + u'_{(\mu}u_{\nu)} - \theta h_{\mu\nu}/3, \qquad \sigma_{\mu\nu}u^\nu = 0$$

and $\theta = u^\mu_{;\mu}$, where u^μ, $w_{\mu\nu}$, θ and $\sigma_{\mu\nu}$ are called acceleration, rotation, expansion and shear respectively, and a semicolon means a covariant derivative [15].

For the model (18), we find

$$\theta = \frac{2}{C_2\tau^{3n}\exp\left\{2m^2\left(\frac{\tau^{4n+2}}{(4n+1)(4n+2)} + C_1\tau\right)\right\}}$$

$$\times\left[\frac{3n}{\tau} + 2m^2\left\{\frac{\tau^{4n+1}}{(4n+1)} + C_1\right\}\right]. \tag{21}$$

The shear scalar σ defined by $\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}$ has the value

$$\sigma = \frac{2m\left(\dfrac{\tau^{4n+1}}{4n+1} + C_1\right)}{\sqrt{3}\left(C_2\tau^{5n}\right)^{\frac{1}{2}} \exp\left\{m^2\left(\dfrac{\tau^{4n+2}}{(4n+1)(4n+2)} + C_1\tau\right)\right\}}, \tag{22}$$

which is non zero for all values of $(0 < \tau < \infty)$ and drops to zero at infinite time $(\tau \to \infty)$. The model (18) has a point singularity at $\tau = 0$ where the pressure and energy density are infinite. As $\tau \to \infty$ the pressure and energy density tends to zero. Therefore the model (18) gives essentially an empty universe for large τ. As $\tau \to \infty$ the ratio $\frac{\sigma}{\theta} \to \frac{1}{\sqrt{3}}$ which shows that the shear scalar does not tend to zero faster than the expansion. The metric (18) represents an expanding and anisotropic cosmological model in which all of the fluids are acceleration-free and rotation-free.

The Hubble parameter H and the deceleration parameter q are respectively given by

$$H = \frac{2}{3}\,\frac{\left[\dfrac{3n}{\tau} + 2m^2\left\{\dfrac{4n+1}{4n+1} + C_1\right\}\right]}{C_2\tau^{3n}\exp\left\{2m^2\left(\dfrac{\tau^{4n+2}}{(4n+1)(4n+2)} + C_1\tau\right)\right\}}, \tag{23}$$

$$q = 1 + \frac{2m^2\tau^{4n} - \dfrac{7n^2}{\tau^2} - \dfrac{2n}{\tau} - \dfrac{10nm^2}{\tau}\left\{\dfrac{\tau^{4n+1}}{4n+1} + C_1\right\} - 4m^4\left\{\dfrac{\tau^{4n+1}}{4n+1} + C_1\right\}^2}{\left[\dfrac{3n}{\tau} + 2m^2\left\{\dfrac{\tau^{4n+1}}{4n+1} + C_1\right\}\right]^2} \tag{24}$$

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